Comparison of Residual and Equilibrated Error Estimators for FEM Applied to Magnetostatic Problems

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***Abstract*—In the computation field domain based on finite element methods, the choice of the mesh is an important step to obtain an accurate solution. In order to evaluate the quality of the mesh, a posteriori error estimators can be used. In this paper we propose to analyze and compare two error estimators for magnetostatic problems.**

I. Introduction

In order to avoid too much computation error in the field analysis based on finite element methods, numerical error estimations are more and more used today.

Some of them are based on an *a priori* error analysis which allows to evaluate the global error before the finite element computation, depending on the exact solution regularity. The others, based on *a posteriori* techniques, give in particular the spatial error distribution which can be used in the remeshing step, by computing estimators depending on the numerical solution.

Many a posteriori error estimators are nowadays available. Among them, some are based on the non-verification property of equilibrium equations (for instance in magnetostatic, Ampere's law and flux conservation [1]), and allow to provide very sharp and global error bounds. Another one recently proposed is based on the energy approach [2]. At last, residual based error estimators that also give rise to local error estimations can be considered [3, 4, 5, 6].

In this communication we propose to develop in the case of magnetostatic problems in term of scalar and vector potentials the residual based error estimators. These estimators will be compared with the equilibrated estimator. To carry out the comparison of all these estimators, a problem with an analytical solution and the classical problem (Team Workshop 13) will be treated.

II. Error Estimators

A. Residual based error estimator

Let be u the exact solution of the problem and u_h the numerical solution, where h characterizes the mesh resolution. Let us call ε the numerical error such that $\varepsilon = \|u - u_h\|$. It can be shown that the error is bounded upper and below [4, 6]:

$$C_1\,\eta \le \varepsilon \le C_2\,\eta, \tag{1}$$

where C_1 and C_2 are positive constants independent of u and h. In this equation η represents the error estimator obtained from the computation. This estimator can be expressed from :

$$\eta^2 = \sum_{T\in\mathscr{T}_h} \eta_T^2, \tag{2}$$

where $\mathscr{T}_h$ is the set of the mesh elements and η_T the local estimator in a given element, which depends on the formulation used.

B. Scalar potential formulation

In the scalar potential formulation the magnetic field $\mathbf{H}$ can be written under the form: $\mathbf{H} = \mathbf{H}_s - \mathbf{grad}\ \Omega$, where $\mathbf{H}_s$ represents the source field and Ω the scalar potential. With this formulation, using the finite element method we want to solve the equation:

$$\mathrm{div}\,(\mu(\mathbf{H}_s - \mathbf{grad}\ \Omega)) = 0, \tag{3}$$

where μ represents the magnetic permittivity. Let be Ω the exact solution and Ω_h the numerical one. The error in scalar potential formulation takes the form:

$$\varepsilon_\Omega^2 = \int_D \mu\big|\mathbf{grad}(\Omega - \Omega_h)\big|^2\, \mathrm{d}\zeta. \tag{4}$$

On the other hand, with the Ω-formulation the local residual estimator in an element η_T can be written under the form [3]:

$$\eta_T^2 = \eta_{T;1}^2 + \eta_{F;1}^2 + \eta_{F;2}^2, \tag{5}$$

where the different estimator contributions are defined by :

$$\begin{aligned}
\eta_{T;1}^2 &= \frac{h_T^2}{\mu_T}\|\mathrm{div}\ (\mu\mathbf{H}_s) - \mathrm{div}\ (\mu\mathbf{grad}\ \Omega)\,\|_{L^2(T)}^2\\
\eta_{F;1}^2 &= \sum_{F\in\mathcal{F}_h^{int}\cap\partial T} \frac{h_F}{2\mu_A}\|[\mu(\mathbf{H}_s - \mathbf{grad}\ \Omega_h)\cdot\mathbf{n}]_F\|_{L^2(F)}^2\\
\eta_{F;2}^2 &= \sum_{F\in\mathcal{F}_h\cap\Gamma_\mathbf{B}} \frac{h_F}{\mu_T}\|\mu\mathbf{H}_s\cdot\mathbf{n} - \mu\mathbf{grad}\ \Omega_h\cdot\mathbf{n}\|_{L^2(F)}^2
\end{aligned}$$

In these definitions, $\mathcal{F}_h^{int}$ is the set of the internal facets of the triangulation. h_T represents the diameter of the smallest sphere containing the element, μ_T the permeability in the element, h_F the diameter of the smallest circle containing the facet and μ_A the maximal value of the permeability of the two elements belonging to the facet. The unit normal to the facet is denoted $\mathbf{n}$, and $[v]_F$ denotes the jump of the quantity v through the facet F. It can be noted that $\eta_{T;1}$ can be equal to zero if $\mathbb{P}_1$ element is used. $\eta_{F;1}$ represents the conservation of the magnetic field at the interface between two elements and $\eta_{F;2}$ controls the boundary condition verification on $\Gamma_\mathbf{B}$.

C. Vector potential formulation

Using the vector potential formulation (**A**-formulation) the magnetic flux density is expressed under the form $\mathbf{B} = \mathbf{curl}\ \mathbf{A}$. With this formulation the equation to solve takes the form:

$$\mathbf{curl}\ \left(\mu^{-1}\mathbf{curl}\ \mathbf{A}\right) = \mathbf{J}_s, \tag{6}$$

where $\mathbf{J}_s$ represents the source current. Similarly to the scalar potential formulation the error can be written under the form:

$$\varepsilon_{\mathbf{A}}^2 = \int_D \mu^{-1}\left|\mathbf{curl}\ (\mathbf{A} - \mathbf{A}_h)\right|^2 \mathrm{d}\zeta, \tag{7}$$

and the local estimator in an element can be expressed by:

$$\eta_T^2 = \eta_{T;1}^2 + \eta_{F;1}^2, \tag{8}$$

where the different estimator contributions are defined by :

$$\begin{aligned}
\eta_{T;1}^2 &= h_T^2 \mu_T \left\|\mathbf{J}_s - \mathbf{curl}(\frac{1}{\mu}\mathbf{curl}\ \mathbf{A}_h)\right\|_{L^2(T)}^2 \\
\eta_{F;1}^2 &= \sum_{F \in \mathcal{F}_h^{int} \cap \partial T} \frac{1}{2} h_F \mu_A \left\|\left[\mathbf{n} \times \frac{1}{\mu}\mathbf{curl}\ \mathbf{A}_h\right]_F\right\|_{L^2(F)}^2.
\end{aligned}$$

In these relations, μ_A is the minimal value of the permeability of the two elements belonging to the facet.

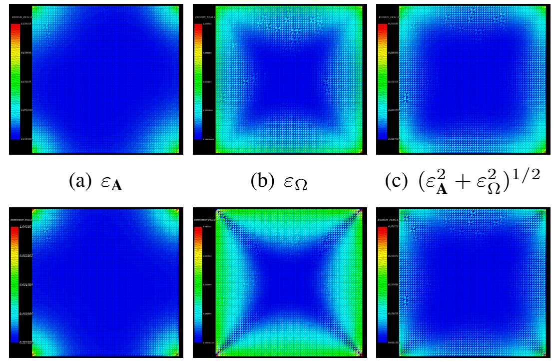
D. Error estimator of the equilibrium method

The error estimator based on the non-verification of the constitutive relationship is also called equilibrated estimator. To compute this estimator, it is necessary to compute two admissible solutions which have to verify the equilibrium equations (3) and (6) called $\mathbf{H}_h$ and $\mathbf{B}_h$ for respectively the Ω and the **A** formulations. From both solutions it is possible to evaluate the local error η_T using the next equation [1]:

$$\eta_T^2 = \|\mathbf{B}_h - \mu\mathbf{H}_h\|_{L^2(T)}^2. \tag{9}$$

III. Numerical Applications

To evaluate the efficiency of all proposed estimators, an academic example based on an analytical solution is studied. The structure is a cube of $1m^3$ crossed by a current density of $10^7 A/m^2$. The analytical solution knowledge allows to evaluate the local exact error map for each formulation, and to compare it to the map provided by the estimators computation. In the same way a comparison between the global errors ε_Ω^2 and $\varepsilon_{\mathbf{A}}^2$ can be carried out. All error maps are displayed in the middle of the cube in a transverse section with regard to the current density direction. In Fig. 1 for the **A**-formulation the exact error map ($\varepsilon_{\mathbf{A}}$) Fig. 1(a) is compared with the error map obtained from the residual estimator Fig. 1(d). For the Ω-formulation the same comparison is carried out between the exact error (ε_Ω) Fig. 1(b) and the residual error estimator Fig. 1(e). The Fig. 1(c) represents $(\varepsilon_{\mathbf{A}}^2 + \varepsilon_\Omega^2)^{1/2}$ that we compare to the equilibrated estimator in its local formulation Fig. 1(f). In order to extend the analysis of these estimators, the Team Workshop 13 has been modeled. The mesh is presented in Fig 2(b). The Fig. 2(a) and Fig. 2(c) represent respectively the residual estimator map ($\eta_{\mathbf{A}}$) and equilibrated estimator map.



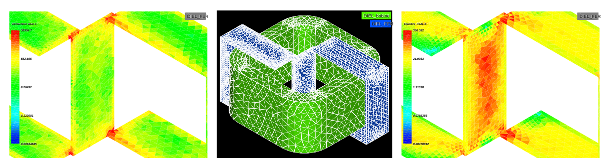
(a) $\varepsilon_{\mathbf{A}}$ (b) ε_Ω (c) $(\varepsilon_{\mathbf{A}}^2 + \varepsilon_\Omega^2)^{1/2}$

(d) $\eta_{\mathbf{A}}$ (e) η_Ω (f) $\eta_{\text{equilibrated}}$

Fig. 1. Local Error Maps

IV. Conclusion

In this communication an analysis of residual and equilibrated estimators has been carried out in the case of magnetostatic formulations. Compared to the exact error, it is clear that both of the residual estimators give an error map in very good accordance with the actual error distribution, each of them for the formulation it is associated with. So they can be successively used for remeshing. The equilibrated estimator gives the error map corresponding to an average value of the errors obtained with the two formulations, and provides the sharper upper bound for the global error [1]. Consequently, the best approach will be to use residual estimators for the local analysis of each formulation, and then the equilibrated estimator to evaluate the global error when needed.



(a) $\eta_{\mathbf{A}}$ (b) mesh (c) $\eta_{\text{equilibrated}}$

Fig. 2. Team Workshop 13

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